Expansion and contraction of avalanches in the two-dimensional Abelian sandpile

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(Received 5 March 1998)

We present a detailed analysis of large scale simulations of avalanches in the two-dimensional Abelian sandpile model. We compare statistical properties of two different decompositions of avalanches into clusters of topplings and waves of topplings. Auxiliary critical exponents are introduced and the existence of the exponent governing the contraction of avalanches claimed in our previous work [Priezzhev *et al.*, Phys. Rev. Lett. **76**, 2093 (1996)] is confirmed. We also give more elaborate argumentation for the exact values of the exponents characterizing the statistics of waves. [S1063-651X(98)05809-7]

PACS number(s): 64.60.Lx

I. INTRODUCTION

The sandpile model introduced by Bak, Tang, and Wiesenfeld [1] serves not only as a lapidary formulation of basic principles of self-organized criticality (SOC) but also seems to be an appropriate candidate for exact determination of all important critical exponents. Indeed, the first steps following Dhar's discovery [2] of the Abelian structure of the sandpile model were encouraging. They include determination of the total number of allowed configurations in the recurrent state [2], evaluation of the height probabilities [3,4] and height-height correlation functions [3,5], and interpretation of the inverse Laplacian operator Δ^{-1} as an expected number of topplings at a given site due to a particle added to another one [2]. Nevertheless, all analytical results obtained up to now catch either static properties of the recurrent state or diffusionlike dynamics of individual particles. The avalanche dynamics as such, responsible for SOC, slips off the analytical description even in the simplified limiting case of large avalanches. The main obstruction is that existing renormalization group methods [6] neglect essential peculiarities of the toppling process, and the complicated spatiotemporal structure of avalanches prevents exact evaluation of the critical exponents.

To advance the analysis of avalanche dynamics, various decompositions of avalanches in the Abelian sandpile model (ASM) into more elementary objects have been proposed. In particular, Grassberger and Manna noticed [7] that each avalanche can be represented as a set of embedded clusters of sites related to a given number of topplings. To make use of this construction for determination of critical exponents it is desirable to obtain a dynamical procedure that naturally divides the avalanche into a collection of clusters. It means that due to the Abelian property of toppling operators, one can try to change the order of topplings so that each avalanche would expand to the largest cluster and then contract by smaller and smaller sets of toppling sites. In our previous works [8–10] we made such an attempt by proposing a decomposition of avalanches into waves of topplings.

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The main feature of the wave structure of avalanches is a possibility to set up a one-to-one correspondence between waves and two-rooted spanning trees [8]. Using the spanning tree representation for waves, one can apply the methods of graph theory to calculate the critical exponents of wave and avalanche statistics [9,10].

On the other hand, it has been found out [11,8] that the set of waves and the set of clusters for a particular avalanche do not coincide. Namely, waves have such an irrelevancy in their superimposing that the next wave can overlap the previous one and the package of waves does not form embedded sets of sites like clusters. The observations of Dhar and Manna [11] and our simulations on small lattices raise hope that the overlappings of waves are relatively rare events. We have assumed [10] that one can neglect the difference between clusters and waves of topplings and consider each next wave embedded into the previous one as a typical situation. Based on this assumption we suggested a method of evaluation of the basic critical exponents of 2D ASM. Our latest simulations, however, have shown that the next wave typically overlaps the previous one. Moreover, the large-scale simulations of Paczuski and Boettcher [12] state that the average difference in size between two subsequent waves is actually negative.

Nevertheless, we will show in this paper that it is possible to modify our simplified scenario of the avalanche process and to describe the phases of expansion and contraction in terms of the wave decomposition. Besides, we will demonstrate that the theoretical predictions are in complete agreement with the numerical data obtained by Paczuski and Boettcher [12].

The paper is organized as follows. In Sec. II we formulate the ASM model, define avalanche clusters and waves of topplings, and introduce the basic ideas of expansion and contraction of avalanches. Section III is devoted to the description of the local dynamics of waves and a proof of the existence of the contraction exponent. In Sec. IV we present analytical derivation of the exponents of conditional distribution of waves obtained by Paczuski and Boettcher [12] from extensive numerical simulations. In Sec. V, an elucidating point of view on the renormalization group approach [6] to the sandpile model is suggested.

II. BASIC CONCEPTS

We recall the definitions of the model, waves of topplings, and avalanche clusters and explain our basic ideas.

In 2D ASM one starts from the empty square lattice (occupation numbers $z_i = 0$ for all sites) and drops sand, particle by particle, at random sites: $z_i \to z_i + 1$. If any $z_i > 4$, the site i is unstable and topples: $z_j \to z_j - \Delta_{ij}$ where Δ is the Laplacian matrix. The toppling at i may cause instability at its nearest neighbors. The subsequent topplings continue until there are no more unstable sites. Then, one adds again a particle at a random site, initiating a new chain of topplings and so on. The process of toppling during each perturbation is called an avalanche and the set of toppled sites forms a compact cluster of all toppled sites.

To obtain the wave decomposition of an avalanche [9], one has to topple all sites that become unstable after adding a particle at i, keeping this site out of the second toppling. The set W_1 of toppled sites is the first wave of topplings. All sites except maybe the site i become stable after the first wave. If the resulting height $z_i > 4$ one topples the site i the second time and continues the relaxation process, not permitting this site to topple a third time. The new set W_2 of relaxed sites is the second wave. The process continues until the site i becomes stable and the avalanche stops.

Grassberger and Manna [7] defined clusters $C_n, n = 1, \dots, M$ of sites toppled not less than n times during the given avalanche, M is the number of topplings at the initially perturbed site. The sets $C_n, n \leq M$ are all compact and each C_n contains the clusters $C_{n+1}, \dots, C_M$.

It is possible to evaluate the asymptotics of cluster size distribution considering the set of generated clusters without reference to a particular avalanche they belong to. According to [2], the expected number of topplings at site j due to adding a particle at site i is given by the lattice Green function $G_{ij} = [\Delta^{-1}]_{ij}$. The number of topplings G_{ij} coincides with the expected number of clusters C_n containing the site j in an avalanche started at i. Therefore, the probability that the linear extent r of a cluster exceeds the distance $|i-j|$ between i and j is

$$\mathrm{Prob}(r > |i-j|) \sim G_{ij}. \tag{1}$$

Using the known asymptotics of the Green function for large distances $G(r) \sim \ln(r)$ and compactness of clusters (cluster area $s_c \sim r^2$), we get

$$P(s_c) = P(r)\frac{dr}{ds_c} \sim \frac{1}{s_c}. \tag{2}$$

It was established in [8] that every wave is a compact set without holes and each site in a wave topples exactly once in that wave. Thus, the expected number of topplings G_{ij} given by the lattice Green function can be expressed alternatively by the probability that a wave taken from an arbitrary avalanche initiated at site i covers site j. Writing Eq. (1) for waves, we get again a size distribution similar to Eq. (2),

$$P(s_w) \sim \frac{1}{s_w}, \tag{3}$$

where s_w is the area of a wave.

To find critical exponents characterizing the size distribution of avalanches, one also needs a general picture of the avalanche process as a whole. In the case of clusters, the picture is quite clear. The set of clusters is ordered and each next cluster is embedded into the previous one. However, the clusters of topplings, being convenient for a computer decomposition of avalanches, are hardly reproducible by dynamical rules as each cluster grows monotonically during the whole avalanche process.

On the contrary, the wave construction admits a simple dynamical interpretation but loses the property of ordering, which is inherent in the case of clusters. In spite of the irregularity of waves, we are still able to use a partial ordering of waves assuming that a typical avalanche consists of two phases: fast expansion and slow contraction. The first phase contains relatively few waves with a large negative difference between subsequent waves $\Delta(s_k) = s_k - s_{k+1}$. The second phase forms the main body of an avalanche with a positive average difference $\Delta(s_k) > 0$. In [10] the fast phase was associated with the single first wave which reaches at once a maximal size the given avalanche spreads. The positive difference for the rest of the waves was assumed to be dependent only on the size of a preceding wave and satisfied the scaling law

$$\langle \Delta(s) \rangle \sim s^{\alpha} \tag{4}$$

for large s.

If the law Eq. (4) is valid for clusters of topplings as well, the density of clusters can be defined as the average number of clusters of size between s_c and $s_c + ds_c$ in one avalanche of the size $S > s_c$:

$$\frac{dn}{ds_c} = \frac{1}{s_c^{\alpha}}. \tag{5}$$

By assumption, the density depends on s_c but not on S. Then, the critical exponent τ in the distribution of a number of sites covered by an avalanche $P(S) \sim S^{-\tau}$ can be related with the exponent α in Eq. (4). Indeed, the probability distribution of cluster sizes $P(s_c)$ is proportional to the probability of avalanches whose size S exceeds s_c: $P(S > s_c) \sim s_c^{-\tau+1}$ and to the density of clusters Eq. (5):

$$P(s_c) \sim s_c^{-\tau+1} s_c^{-\alpha}. \tag{6}$$

Comparing Eq. (6) with Eq. (2) we obtain

$$\tau + \alpha = 2, \tag{7}$$

and the problem of finding the basic exponent τ is reduced to a search for the exponent α, which is related more directly to details of the avalanche process.

The "contraction" exponent α is well defined for avalanche clusters or for waves provided that one can neglect the differences between these two kinds of objects. In this connection, the following questions arise. Is it possible to define the "contraction" exponent for waves taking into account overlappings and, if so, what is the correspondence between its numerical values for clusters and waves? Can we

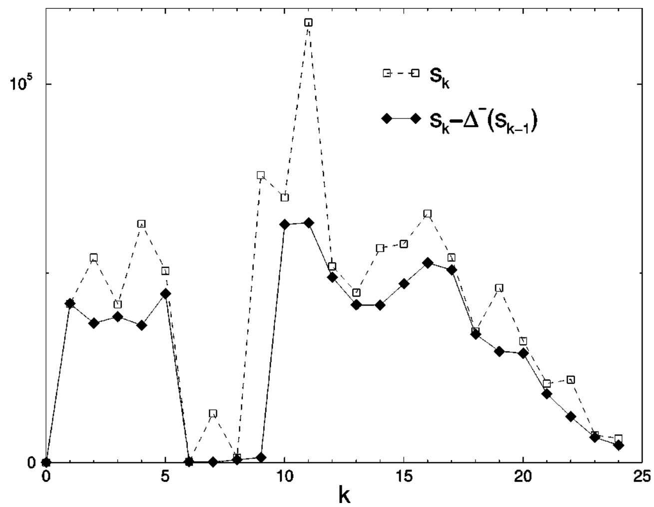

FIG. 1. Sizes s_k of waves in a typical avalanche on the lattice of size L^2, $L=500$ (empty squares); the same quantities subtracted by the size of overlapping: $s_k-\Delta^-(s_{k-1})$ (filled diamonds).

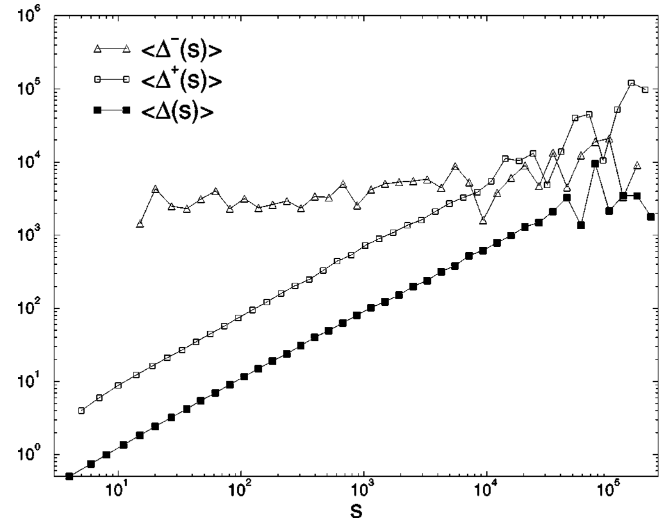

FIG. 2. The average values of $\Delta^-(s),\Delta^+(s)$ for waves and $\Delta(s)$ for clusters as functions of their size s (see text for definitions), obtained from the simulations data of 10^6 avalanches on lattice of size $L=500$. The graph $\Delta(s)$ is shifted vertically.

establish the same relation Eq. (7) for waves and use their spanning tree structure to estimate the critical exponents of the model?

In the following sections we discuss these questions using large scale simulations of clusters of topplings, a more elaborate analysis of waves, and numerical data for subsequent waves obtained by Paczuski and Boettcher [12].

III. CONTRACTION EXPONENT FOR WAVES

First, we present a picture of a typical avalanche of 2D ASM on a square lattice of size L^2 for $L=500$. In Fig. 1 we plot the size s_k of the wave as a function of its number k in the avalanche. We can see that many of the next waves have a size greater than the size of the previous one. Moreover, even those waves, whose size is actually less than the size of its predecessor, are not most frequently embedded into the set of sites formed by the previous wave. For the particular avalanche presented in Fig. 1 the event of overlapping the previous wave by the next one occurs for all waves except the sixth and the last one.

Anyway, one can note that a typical avalanche contains several sharp peaks corresponding to fast expansion of the avalanche size and in-between intervals of relatively slow, although irregular, contraction. One may expect that the average difference between subsequent waves in the slow phase follows a scaling law similar to Eq. (4). To verify this, one has to extract from the averaging of $\Delta(s_k)$ those waves that are related to the expansion phase. We can avoid this cumbersome and ambiguous procedure by introducing new variables characterizing the ''local'' contraction and expansion.

Consider two typical subsequent waves of topplings W_k and W_{k+1} with the sizes s_k and s_{k+1}, the $(k+1)$st wave overlaps the kth wave. Let W be their intersection having size s.

Define the variables $\Delta^+(s_k)=s_k-s$ and $\Delta^-(s_k)=s_{k+1}-s$; the first quantity is ''local contraction,'' the second one refers to ''local expansion.'' We calculated the averages $\langle\Delta(s_k)\rangle$ for clusters, $\langle\Delta^+(s_k)\rangle$ and $\langle\Delta^-(s_k)\rangle$, for waves of topplings using data of 10^6 avalanches for the system size $L=500$ (Fig. 2). The simulations show a power-law behavior $\langle\Delta(s_k)\rangle\sim s^\alpha$ for clusters and $\langle\Delta^+(s_k)\rangle\sim s^{\alpha^+}$ for waves; the exponents α and α^+ have close values. The value of the exponent α^-, for the relation $\langle\Delta^-(s_k)\rangle\sim s^{\alpha^-}$, is much smaller than α^+.

Concerning the estimation of the exponents α, α^+, and α^-, we have to point out that the numerical determination of these values is a rather difficult problem because of a slow convergence of data obtained for large lattices to their limiting values. So, our numerical results $\alpha\approx\alpha^+\approx0.88$ and $\alpha^-\approx0.29$ for $L=500$ are still far from the expected limit. The problem of estimation of these exponents is somehow similar to the numerical determination of the exponent τ (for discussion, see, for example, [13]). The extrapolation $L\to\infty$ gives us some wide interval of possible values of α and α^+ that includes 3/4, the theoretical prediction [10] for the exact value.

Being equivalent to τ from a computational point of view, the exponent α^+ is more convenient for theoretical evaluations. The spanning tree representation of waves [8] makes it possible to interpret $\Delta^+(s)$ as a sum of branches attached to the boundary of a wave [10], and then to use exact results obtained for the q-component Potts model [14] in the limit $q\to0$.

The relative magnitude of $\Delta^-(s_k)$ and $\Delta^+(s_k)$ is such that for large s the contraction of avalanche dominates its expansion. We show in Fig. 1 by filled diamonds the sizes of waves s_k subtracted by the size $\Delta^-(s_{k-1})$. It is clear that neglecting the quantities $\Delta^-(s_k)$ we do not change the qualitative dynamical picture of the avalanche and the contraction of waves can be described in terms of $\Delta^+(s_k)$. Based on these data we can also see that the average $\langle\Delta(s_w)\rangle$ for waves, which is equal to the remainder $\langle\Delta^+(s_k)\rangle-\langle\Delta^-(s_k)\rangle$, is actually negative for small waves and positive for large ones, as found in [12].

Finally, we establish the relationship of the same type as Eq. (7) for the exponents τ and α^+. Following argumentation for clusters (Sec. II), we estimate the known asymptotics

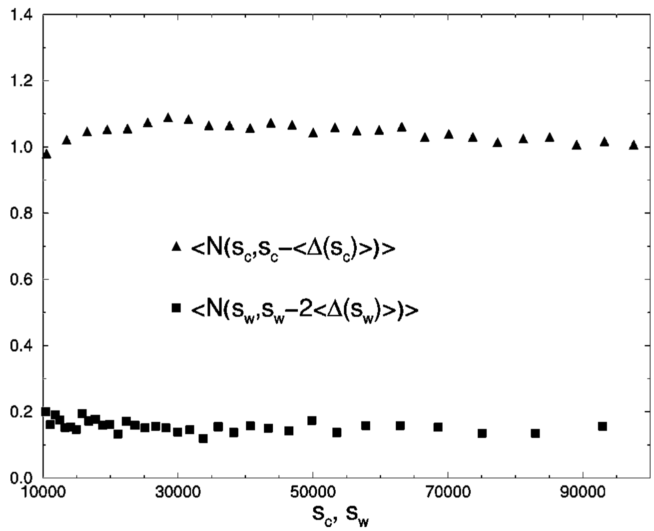

FIG. 3. The expected number $\langle N(s,s-t)\rangle$ of clusters (waves) of size from the interval $[s-t,s]$ provided that the size of the corresponding avalanche is greater than s. Due to fluctuations the data for waves are uncertain for some large values of s_w, so the averages are not proportional.

of size distribution $P(s_w)$ of waves Eq. (3), which is proportional to the probability of avalanches whose size S exceeds s_w, $P(S>s_w)\sim s_w^{-\tau+1}$, and to the density of waves

$$P(s_w)\sim s_w^{-\tau+1}\frac{dn}{ds_w}. \tag{8}$$

Let $N(s,s-t)$ be the number of waves in a particular avalanche with sizes between $s-t$ and s provided that the size of the given avalanche is greater than s. The asymptotic behavior of dn/ds_w can be evaluated as

$$\frac{dn}{ds_w}\sim\frac{\langle N(s_w,s_w-t)\rangle}{t} \tag{9}$$

for large $s_w \gg t$. Take in Eq. (9) $c\langle\Delta^+(s_w)\rangle$ instead of t where $c=O(1)$ is a constant. Then, from Eq. (8) we obtain

$$s_w^{-1}\sim s_w^{-\tau-\alpha^+ +1}\langle N(s_w,s_w-c\langle\Delta^+(s_w)\rangle)\rangle. \tag{10}$$

In Fig. 3 we present the quantity

$$\langle N(s_w,s_w-2\langle\Delta^+(s_w)\rangle)\rangle$$

as a function of s_w calculated from our numerical data where we put $c=2$. It is apparently evident that asymptotically it scales as $O(1)$. Thus, we obtain from Eq. (10)

$$\tau+\alpha^+=2. \tag{11}$$

For comparison in Fig. 3 we present the function

$$\langle N(s_c,s_c-\langle\Delta(s_c)\rangle)\rangle$$

for clusters, which also scales as $O(1)$, confirming Eq. (7).

IV. ANALYSIS OF CONDITIONAL WAVE DISTRIBUTION

Recently, Paczuski and Boettcher [12] have undertaken careful numerical simulations to find the size distribution of subsequent waves for a given size of the preceding wave $P(s_{k+1}|s_k)$. The data were represented by the scaling form

$$P(s_{k+1}|s_k)\sim s_{k+1}^{-\beta}F\left(\frac{s_{k+1}}{s_k}\right), \tag{12}$$

where $F(x)\rightarrow$const when $x\rightarrow 0$ and $F(x\gg 1)\sim x^{-r}$. The function $P(s_{k+1}|s_k)$ being considered as a normalized distribution of s_{k+1} should be multiplied by the factor $s_k^{\beta-1}$ to provide the normalization condition

$$\int^{s_{\rm co}}P_N(s_{k+1}|s_k)ds_{k+1}=\int^{s_{\rm co}/s_k}x^{-\beta}F(x)dx, \tag{13}$$

where $s_{\rm co}\sim L^2$ is the cutoff in the wave sizes from the finite system size [12]. The normalized function $P_N(s_{k+1}|s_k)$ has the asymptotics

$$P_N(s_{k+1}|s_k)\sim s_{k+1}^{-\beta}s_k^{\beta-1} \tag{14}$$

when $s_{k+1}\ll s_k$, and

$$P_N(s_{k+1}|s_k)\sim s_{k+1}^{-\beta-r}s_k^{\beta+r-1} \tag{15}$$

when $s_{k+1}\gg s_k$.

To be consistent with the analysis in [10], both exponents β and r should be explained from the same point of view based on the spanning tree representation of waves. Before doing that, we will discuss an attempt to verify the existence of the scaling law Eq. (4) for the contraction phase of an avalanche by calculating the average difference between subsequent waves $\langle\Delta(s_k)\rangle=\langle s_k-s_{k+1}\rangle$ [12]. By the assumption [10], the average $\langle\Delta(s_k)\rangle\sim s_k^{\alpha}>0$ for the main part of an avalanche, corresponding to the process of slow contraction of wave fronts. A serious problem, however, is how to select the waves relating to the slow phase. In a real avalanche, at least waves corresponding to the largest contribution to the expansion should be removed from the averaging as explained in Sec. II. Without the selection of waves responsible for the contraction of avalanches the result of averaging of Δs obtained in [12] is easily predictable.

Following [12], fix a value s and take all waves with $s_k=s$ together with the subsequent waves of size s_{k+1} from all avalanches whose size $S\geq s$. Consider separately the cases $s_{k+1}<s_k$ and $s_{k+1}>s_k$. In the first case, the average difference $\langle\Delta s^{(-)}\rangle=\langle s_k-s_{k+1}\rangle$ is obviously $\langle\Delta s^{(-)}\rangle<s$.

In the opposite case, the waves with $s_{k+1}>s_k$ have a power-law asymptotics $P(s_{k+1})\sim s_{k+1}^{-\theta}$ where $1<\theta<2$ for all s_k. Indeed, the size distribution of waves with an arbitrary origin is $P(s)\sim s^{-1}$ according to the two-component spanning tree representation [9]. The distribution of waves of size s with the origin in a fixed unique site is $P'(s)\sim s^{-2}$. In the considered case, the subsequent wave starts at a site in the localized area inside the previous wave. This implies $1<\theta<2$. Therefore, the averaged positive difference $\langle\Delta s^{(+)}\rangle=\langle s_k-s_{k+1}\rangle\sim L^{2(2-\theta)}-s$ and diverges with the lattice size L. Thus, $\langle\Delta s^{(+)}\rangle>\langle\Delta s^{(-)}\rangle$ until s becomes large and finite-size effects become essential. We see that the negative values of $\langle\Delta s(s_k)\rangle$ obtained in [12] are not actually surprising and indicate only that the simple average $\langle\Delta s\rangle$ does not exhibit a power-law dependence on s_k and cannot be related directly to the density of waves.

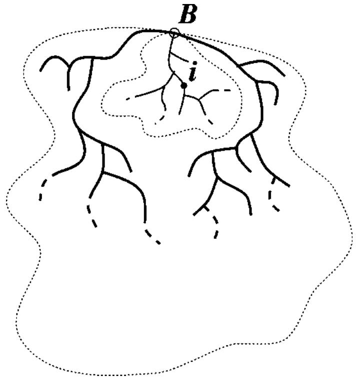

FIG. 4. The structure of spanning trees representing two subsequent waves with sizes $s_{k+1}<s_k$. The origin of the avalanche is denoted by i. B is the point of intersection of the boundaries of these waves.

Nevertheless, the distribution Eq. (12) itself brings important information about avalanches. The exponents characterizing its asymptotics are related to basic exponents of the sandpile model. Below, we present qualitative arguments leading to estimations of their numerical values.

Consider first the exponent β. According to Eq. (12), this exponent determines the behavior of smaller waves following just after waves of larger sizes: $s_{k+1}\ll s_k$. The wave W_k can be represented by a tree T_k covering the area s_k and having the root i at the point where the wave W_k was initiated [9]. The tree T_{k+1} representing the wave W_{k+1} has the root at the same point i.

To provide the sharp reduction of the next wave, T_k and T_{k+1} must have a special structure. Since each site in a wave topples exactly once, the state of the system inside a wave does not change after the wave is completed. This means that any branch of T_{k+1} attached to the root i repeats a branch of T_k until it ends inside s_k or touches the boundary of s_k. It follows from $s_{k+1}\ll s_k$ that at least one point B exists where the boundaries of W_k and W_{k+1} intersect (Fig. 4). In the vicinity of B, both trees T_k and T_{k+1} can be attached by a bond b to their complemented subtrees T'_k and T'_{k+1} [9] defined by the condition that a unification T and T' gives a complete spanning tree of the whole lattice. Inversely, deletion of the bond b from the corresponding trees produces subtrees T_k and T_{k+1}, which obey the statistics of disconnected branches of lattice spanning trees [15] or, equivalently, the asymptotics of last waves [8]

$$P_{\mathrm{last}}(s)\sim\frac{1}{s^{11/8}}. \tag{16}$$

It has been demonstrated in [12] that the function $F(x)$ in Eq. (12) is constant in a finite interval $0<x<c<1$. Thus, we can consider the distribution

$$P_c(s_{k+1})=\int_{s_k>s_{k+1}/c}P_N(s_{k+1}|s_k)\,ds_k \tag{17}$$

with the function $P_N(s_{k+1}|s_k)$ taken in the form Eq. (14). This gives

$$P_c(s_{k+1})=s_{k+1}^{-\beta}\int_{s_k>s_{k+1}/c}s_k^{\beta-1}ds_k\sim\frac{L^{2\beta}}{s_{k+1}^{\beta}}. \tag{18}$$

On the other hand, $P_c(s_{k+1})$ apart from the normalization factor is given by a joined probability distribution of disconnected branches T_k and T_{k+1}. Despite the fact that subtrees T_k and T_{k+1} are strongly connected (T_{k+1} is a part of T_k), the distributions of their sizes can roughly be considered as independent. Then, we obtain

$$P_{\mathrm{last}}(s_{k+1})P_{\mathrm{last}}(s_k>s_{k+1}/c)=s_{k+1}^{-11/8}\int_{s_k>s_{k+1}/c}s_k^{-11/8}ds_k$$

$$\sim\frac{1}{s_{k+1}^{7/4}}. \tag{19}$$

To get $P_c(s_{k+1})$ from Eq. (19), we have to multiply the last expression by s_{k+1}, the number of possible positions of the root i inside the disconnected branch T_{k+1}. Finally, $P_c(s_{k+1})\sim 1/s_{k+1}^{3/4}$ and comparing with Eq. (18) we get $\beta=3/4$, which explains the numerical result [12].

To relate the exponent r with the exponent τ in the distribution of a number of distinct sites covered by an avalanche, we consider waves of two types. A wave will be referred to as the growing one or the G wave if $s_{i+1}\geqslant s_i$ and the reducing one or the R wave if $s_{i+1}<s_i$. Every avalanche corresponds to a unique sequence of G waves and R waves, e.g., $GRGGR$, ...

The number of distinct sites s_d in an avalanche is proportional to the size of the maximal wave $W_{\max}$, so we can expect that

$$P(s_{\max})\sim\frac{1}{s_{\max}^{\tau}} \tag{20}$$

with the same critical exponent τ.

The expected number of waves in an avalanche diverges logarithmically with the lattice size [3]. However, if the idea about the fast expansion phase is correct, the expected number of waves in the interval between the maximal wave and the latest wave W_{k_0} with $s_{k_0}\sim O(1)$ before the maximal wave should be finite when $L\to\infty$.

Starting with this assumption, consider a finite sequence of n G waves and R waves between the waves W_{k_0+1} and $W_{k_0+m}=W_{\max}$ (for simplicity, we denote their sizes by $s_1,\ldots,s_n$). The first and the last waves in the sequence are clearly of type G. It follows from the numerical data of [12] that the asymptotics Eqs. (14) and (15) of the distribution function $P_N(s_{k+1}|s_k)$ are factorized. Extrapolating the distribution Eq. (15) to the case of vanishing previous waves $s_{k_0}\to 1$, we can obtain the distribution of first waves in the sequence:

$$P(s_1)\sim\frac{1}{s_1^{\beta+r}}. \tag{21}$$

For an avalanche GG, . . . beginning from two G waves, the distribution of the second wave is given by

$$P(s_2)=\int^{s_2} ds_1 P(s_1) P_N(s_2|s_1). \tag{22}$$

Using Eqs. (15) and (21) we have for large s_2

$$P(s_2)\sim \frac{\ln s_2}{s_2^{\beta+r}}. \tag{23}$$

Similarly, the leading asymptotics for the nth wave in a sequence of n G waves is

$$P(s_n)\sim \frac{(\ln s_n)^{n-1}}{s_n^{\beta+r}}. \tag{24}$$

The presence of R waves reduces the logarithmic divergence of the numerator. For instance, using Eq. (14) and Eq. (15) we get in the case $GRG, \ldots$ the numerator $\ln s_3$ instead of $(\ln s_3)^2$ in the case $GGG, \ldots$. Generally, if k $(k \leqslant n-2)$ last G waves in the sequence follow R wave, the convolution

$$P(s_n)=\int \cdots \int ds_1 \cdots ds_{n-1} P(s_1) \times P_N(s_2|s_1) \cdots P_N(s_n|s_{n-1}) \tag{25}$$

has the asymptotics

$$P(s_n)\sim \frac{(\ln s_n)^{k}}{s_n^{\beta+r}}. \tag{26}$$

Thus, for any finite sequence of G waves and R waves between the relatively small (k_0)th wave and the maximal wave, we have the leading exponent $\beta+r$, which governs the distribution of the maximal waves $P_{\max}\sim s_{\max}^{-\tau}\sim s_{\max}^{-\beta-r}$. The numerical values obtained in [12] are $\beta=3/4, r=1/2$. This gives $\tau=5/4$ obtained in [10] from scaling arguments.

V. DISCUSSION

In conclusion, the analysis of the decomposition of avalanches into waves of topplings shows that a difference between two subsequent waves can be described by appropriate variables that follow a power-law dependence on the wave size s. The exponent α^+ corresponding to the contraction of waves can be related to one of the basic avalanche exponents τ.

The relation between the asymptotics Eq. (15) of the distribution of subsequent waves $P_N(s_{k+1}|s_k)$ in the case $s_{k+1} \gg s_k$ and the exponent τ in the distribution of distinct sites involved in an avalanche implies an alternative way of determining τ. Instead of derivation τ from the analysis of slow contraction process, we can use the statistics of large waves W_{k+1} overlapping their predecessors W_k to link τ with the exponents β and r. This approach sheds new light on the renormalization group (RG) procedure proposed by Pietronero *et al.* [6] for the sandpile model. In the RG method, one deals with sites of three classes: stable, critical, and unstable. Extending the characterization of the stationary properties at a generic scale, one describes the propagation of instability through the lattice taking into account only one-shot relaxation events at each scale. Thus, proliferation effects due to multiple relaxations are not considered in this scheme. In this respect, the process described by RG is not a true avalanche, rather it is a wave propagating from a given point or from a cluster of a given size. Correspondingly, the critical exponent determined in this way is actually the sum of exponents $\beta+r$ in the asymptotics of distribution of large waves. Its numerical value 1.248 obtained in [16] is in excellent agreement with the value $\beta+r=5/4$ proposed in [12]. On the other hand, it was shown in Sec. IV that $\beta+r=\tau$, which explains the validity of the RG approach despite the neglect of multiple relaxations.

ACKNOWLEDGMENTS

D.V.K. gratefully acknowledges the financial support from Alexander von Humboldt Foundation. This work was partially supported by the Russian Foundation for Basic Research through Grant No. 97-01-01030. V.B.P. appreciates the hospitality of the Computational Physics Group at Duisburg University.

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